Introduction to the Standard Model
William and Mary PHYS 771 Spring 2014

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(Dated: April 30, 2014 16:40)

Class information, including syllabus and homework assignments can be found at
`http://ntc0.lbl.gov/~walkloud/wm/courses/PHYS_771/`
or
`http://cyclades.physics.wm.edu/~walkloud/wm/PHYS_771/`

Homework Assignment 4: due Wednesday 30 April

1. We discussed the QCD beta function, which at one-loop gives the strong coupling

$$\begin{aligned}\alpha_S(\mu) &= \frac{\alpha_S(Q_0)}{1 + \frac{\alpha_S(Q_0)}{4\pi}\beta_0 \ln\left(\frac{\mu^2}{Q_0^2}\right)} \\ &= \frac{1}{\frac{\beta_0}{4\pi}\ln\left(\frac{\mu^2}{\Lambda_{QCD}^2}\right)}\end{aligned} \tag{1}$$

where

$$\beta_0(N_c = 3) = \frac{33 - 2n_f}{3}\,, \qquad \text{and} \qquad \Lambda_{QCD}^2 = Q_0^2 \exp\left\{-\frac{4\pi}{\alpha_S(Q_0)\beta_0}\right\}\,. \tag{2}$$

In these expressions, n_f is the number of "active" quark flavors, meaning quarks with $m_q < \mu$. Even for massless quarks, QCD dynamically generates an energy scale, Λ_{QCD}, which is known as "Dimensional Transmutation". This scale is simply defined at a given order in perturbation theory as the scale where the coupling diverges. Before reaching this scale, of course the theory becomes non-perturbative, and so this scale provides only a qualitative understanding of the "scale of QCD". Qualitatively, hadrons comprised of u, d and s quarks, whose mass is not protected by chiral symmetry (the pions etc.), have a mass proportional to Λ_{QCD} with corrections from the light quark masses

$$m_H = c_H \Lambda_{QCD} + \mathcal{O}(m_q) \tag{3}$$

which is why the proton is $m_p \sim 1$ GeV.

(a) Fix α_S at the Z-pole. Using the one-loop running, what is $\Lambda_{QCD} =$?

To answer this question, you need to start with the first line of Eq. (1), and run the scale μ down through the heavy quark thresholds. At $\mu = m_q$, you match the the coupling above and below m_q where the running uses different number of "active" flavors above and below the scale $\mu = m_q$. e.g. below M_Z but above m_b, you have 5 active flavors while for $\mu < m_b$, there are only 4 active flavors. Perform this matching and running until you find a scale at which the coupling diverges.

(b) What would be the value Λ_{QCD} if $m_b > M_Z$?

(c) What would be the value Λ_{QCD} if $m_b = 50$ GeV?

(d) if the b-quark mass were to increase, would the mass of the proton increase or decrease? (explain)

2. Cottingham's Formula and the electron electromagnetic self-energy. In class, we discussed the Cottingham Formula and the nucleon electromagnetic self-energy. Here, we will use it to determine the electron self-energy. Cottingham's Formula is

$$\delta M^\gamma = \frac{i}{2M}\frac{\alpha_{f.s.}}{(2\pi)^3}\int_R d^4q \frac{g^{\mu\nu}T_{\mu\nu}(q^0, -q^2)}{q^2 + i\epsilon} \tag{4a}$$

$$= \frac{\alpha_{f.s.}}{8M\pi^2}\int_R dQ^2 \int_{-Q}^{+Q} d\nu \frac{\sqrt{Q^2-\nu^2}}{Q^2} T^\mu_\mu(i\nu, Q^2) \tag{4b}$$

where the subscript R reminds us the integral must be renormalized.

(a) Derive Eq. (4b) from Eq. (4a).

(b) Starting from

$$T_{\mu\nu}(q^0, -q^2) = \frac{i}{2}\sum_\sigma \int d^4\xi e^{iq\cdot\xi}\langle p, \sigma|T\{J_\mu(\xi), J_\nu(0)\}|p, \sigma\rangle \tag{5}$$

this forward Compton Amplitude is crossing symmetric,

$$T_{\nu\mu}(-q^0, q^2) = T_{\mu\nu}(q^0, q^2): \tag{6}$$

Show this to be true.

(c) Use this crossing symmetry to show the scalar functions satisfy

$$T_i(-q^0, -q^2) = T_i(q^0, -q^2) \tag{7}$$

where the $T_i(q^0, -q^2)$ are defined for example below in Eq. (8).

(d) At leading order in QED, what is the electron forward Compton Scattering Amplitude?

i. using the parameterization,

$$T_{\mu\nu} = -\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)T_1(q^0, -q^2) + \frac{1}{M^2}\left(p_\mu - q_\mu\frac{p\cdot q}{q^2}\right)\left(p_\nu - q_\nu\frac{p\cdot q}{q^2}\right)T_2(q^0, -q^2) \tag{8}$$

what are the scalar functions $T_i(q^0, -q^2) = ?$

ii. using the parameterization,

$$T_{\mu\nu} = \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)q^2 t_1(q^0, -q^2) - \frac{1}{M^2}\left(p_\mu p_\nu - \frac{p\cdot q}{q^2}(p_\mu q_\nu + p_\nu q_\mu) + \frac{(p\cdot q)^2}{q^2}g_{\mu\nu}\right)q^2 t_2(q^0, -q^2) \tag{9}$$

A. what is the relation between t_i and T_i?

B. what are the scalar functions $t_i(q^0, -q^2) =$?

(e) Using your determination of the forward Compton Amplitude, evaluate the self-energy in Eq. (4b). To perform this evaluation, use Pauli-Villars with a Q^2 cut-off. Recall, Pauli-Villars replaces the photon propagator with the difference between the photon and a heavy photon. In our Eq. (4b), this amounts to

$$\frac{1}{Q^2} \to \frac{1}{Q^2} - \frac{1}{Q^2 + \Lambda^2} \tag{10}$$

and to make the Q^2 integral finite, we can put in a UV cutoff and add a counterterm, such that our mass self-energy correction becomes

$$\delta M^\gamma = \lim_{Q_{UV} \to \infty} \left[\frac{\alpha_{f.s.}}{8M\pi^2} \int_0^{Q^2_{UV}} dQ^2 \int_{-Q}^{+Q} d\nu \sqrt{Q^2 - \nu^2} T^\mu_\mu(i\nu, Q^2) \left[\frac{1}{Q^2} - \frac{1}{Q^2 + \Lambda^2} \right] + \delta M(\Lambda) \right] \tag{11}$$

where $\delta M(\Lambda)$ is the counterterm needed to render the answer independent of Λ.

i. Evaluate the integral and take the large-Λ limit. What is the resulting expression for the self-energy correction including the finite and logarithmic terms?

ii. Use the ideas of renormalization to determine the counterterm (demand the entire answer be independent of Λ, usually done by taking $\partial/\partial \ln(\Lambda^2) \delta M^\gamma = 0$).

iii. How does your answer compare with the answer using dimensional regularization? (or compare with the answer in the literature/books)